

Yang-Mills flows on nearly Kähler manifolds and G_2 -instantons

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Abstract

We consider $\text{Lie}(G)$ -valued G -invariant connections on bundles over spaces G/H , $\mathbb{R} \times G/H$ and $\mathbb{R}^2 \times G/H$, where G/H is a compact nearly Kähler six-dimensional homogeneous space, and the manifolds $\mathbb{R} \times G/H$ and $\mathbb{R}^2 \times G/H$ carry G_2 - and $\text{Spin}(7)$ -structures, respectively. By making a G -invariant ansatz, Yang-Mills theory with torsion on $\mathbb{R} \times G/H$ is reduced to Newtonian mechanics of a particle moving in a plane with a quartic potential. For particular values of the torsion, we find explicit particle trajectories, which obey first-order gradient or hamiltonian flow equations. In two cases, these solutions correspond to anti-self-dual instantons associated with one of two G_2 -structures on $\mathbb{R} \times G/H$. It is shown that both G_2 -instanton equations can be obtained from a single $\text{Spin}(7)$ -instanton equation on $\mathbb{R}^2 \times G/H$.

1 Introduction and summary

The Yang-Mills equations in two, three and four dimensions have been intensively studied both in physics and mathematics. In mathematics, this study (i.e. projectively flat unitary connections and stable bundles in $d=2$ [1], the Chern-Simons model and knot theory in $d=3$, instantons and Donaldson invariants [2] in $d=4$) has yielded a lot of new results in differential and algebraic geometry. In particular, a crucial role in $d=4$ gauge theory is played by the first-order anti-self-duality equations, which on manifolds $\mathbb{R} \times X^3$ are precisely the Chern-Simons gradient flow equations. The program of extending familiar constructions in gauge theory, associated to problems in low-dimensional topology, to higher dimensions, was proposed in [3] and developed in [4, 5, 6, 7, 8, 9, 10].¹ An important role in this investigation is played by first-order gauge equations which are a generalisation of the anti-self-duality equations in $d=4$ to higher-dimensional manifolds with special holonomy (or, more generally, with G -structure [11, 12]). Such equations in $d>4$ dimensions were first introduced in [13] and further considered e.g. in [14, 15, 16, 17, 6, 10, 18]. Some of their solutions were found e.g. in [19, 20, 21].

In physics, interest in Yang-Mills theories in dimensions greater than four grew essentially after the discovery of superstring theory, which contains supersymmetric Yang-Mills in the low-energy limit in the presence of D-branes as well as in the heterotic case. In particular, heterotic strings yield $d=10$ heterotic supergravity, which contains the $\mathcal{N}=1$ supersymmetric Yang-Mills model as a subsector [22]. Supersymmetry-preserving compactifications on spacetimes $M_{10-d} \times X^d$ with further reduction to M_{10-d} impose the above-mentioned first-order BPS-type gauge equations on X^d [13, 22]. Initial choices for the internal manifold X^d were Kähler coset spaces and Calabi-Yau manifolds, as well as manifolds with exceptional holonomy group G_2 for $d=7$ and $\text{Spin}(7)$ for $d=8$. However, it was realised that Calabi-Yau compactifications suffer from the presence of many massless moduli fields in the resulting four-dimensional effective theories.² This problem can be cured (at least partially) by allowing for non-trivial p -form fluxes on X^d . String vacua with p -form fields along the extra dimensions (‘flux compactifications’) have been intensively studied in recent years (see e.g. [23] for reviews, and also the references therein).

Compactifications in the presence of fluxes can be described in the language of G -structures on d -dimensional manifolds X^d : $\text{SU}(3)$ -structure for dimension $d=6$, G_2 -structure for $d=7$ and $\text{Spin}(7)$ -structure for $d=8$. In the definition of all these G -structures there enters a $(d-4)$ -form Ψ on X^d . Thus, we deal with internal manifolds of special geometry and consider the three-form field $\mathcal{H} = *\text{d}\Psi$ as torsion, where $*$ denotes the Hodge star operator. In particular, in six dimensions these manifolds may be non-Kähler and sometimes even non-complex.

Flux compactifications have been investigated primarily for type II strings and to a lesser extent in the heterotic theories, despite their long history [24]. The number of torsionful geometries that can serve as a background for heterotic string compactifications seems rather limited. Among them there are six-dimensional nilmanifolds, solvmanifolds, nearly Kähler and nearly Calabi-Yau coset spaces. The last two kinds of manifolds carry a natural almost complex structure which is not integrable (for a discussion of their geometry see e.g. [25, 26, 27, 28, 29] and references therein).

In heterotic string compactifications one has the freedom to choose a gauge bundle since the simple embedding of the spin connection into the gauge connection is ruled out for compactifications with $\text{d}\mathcal{H} \neq 0$. For the torsionful backgrounds, the allowed gauge bundle is restricted by the

¹For more literature see references therein.

²Kähler cosets also lead to non-realistic effective theories.

Bianchi identity for the torsion field (anomaly cancellation) and by the Donaldson-Uhlenbeck-Yau equations [15] for $d=6$ or the G_2 -instanton equations [3] for $d=7$. The construction of such vector bundles over G_2 -manifolds of topology $\mathbb{R} \times X^6$ is the subject of the present paper.

The only known examples of compact nearly Kähler six-manifolds are the four coset spaces $SU(3)/U(1) \times U(1)$, $Sp(2)/Sp(1) \times U(1)$, $G_2/SU(3) = S^6$ and $SU(2)^3/SU(2) = S^3 \times S^3$. On all four cosets G/H we have a torsion $\mathcal{H} = *d\omega$ for an almost Kähler form ω . We describe some solutions of the Donaldson-Uhlenbeck-Yau equations for the gauge group G on these cosets. Our ansatz for a G -invariant connection is parameterised by a complex number ϕ , and the solutions show the 3-symmetry characteristic of all nearly Kähler spaces.

Next, we step up to seven dimensions, extending G/H by a real line \mathbb{R}_τ , so that $\phi \rightarrow \phi(\tau) \in \mathbb{C}$ in our G -invariant ansatz. For the torsion $\mathcal{H} = -\frac{1}{3}\kappa_1 * (d\tau \wedge d\omega) + \frac{1}{3}\kappa_2 d\omega$ with $\kappa_1, \kappa_2 \in \mathbb{R}$, our ansatz reduces the Yang-Mills equations to Newton's equations $\ddot{\phi} = f(\phi)$ for a particle in the complex ϕ plane, subject to a 3-symmetric cubic force f . For $\kappa_2=0$, there exists a potential of ϕ^4 type, so $f \sim \frac{\partial V}{\partial \phi}$, and an action can be formulated, which surprisingly agrees with the torsionful Yang-Mills action on our ansatz. Yet, even for $\kappa_2 \neq 0$, we construct an explicit solution.

In special instances, $\ddot{\phi} \sim \frac{\partial V}{\partial \phi}$ is implied by a flow equation $\dot{\phi} \sim \frac{\partial W}{\partial \phi}$. This flow is gradient or hamiltonian, depending on whether the proportionality is real or imaginary. Among the complex $\phi(\tau)$ trajectories, finite-action kinks occur when $(\kappa_1, \kappa_2) = (\pm 3, 0)$ and $(-1, 0)$, as solutions to the gradient and hamiltonian flow, respectively. The corresponding connections are finite-action solutions of the Yang-Mills equations on $\mathbb{R} \times G/H$. By a duality transformation, which relates solutions for different values of κ_1 , infinite-action solutions to the Yang-Mills equations are presented as well.

The cases $(\kappa_1, \kappa_2) = (3, 0)$ and $(-1, 0)$ mentioned above have a clear geometrical meaning. The corresponding gradient and hamiltonian flow equations for ϕ follow from seven-dimensional anti-self-duality conditions based on one of two G_2 -structures, called the G_2 -instanton equations. Now, these both descend from anti-self-duality equations based on the $Spin(7)$ -structure of the eight-dimensional space $\mathbb{R}_\tau \times \mathbb{R}_\sigma \times G/H$. For the gradient case one reduces over \mathbb{R}_σ , while the hamiltonian case arises upon reduction over \mathbb{R}_τ . The G_2 -instanton equations can themselves be interpreted as gradient and hamiltonian flows for a certain action functional on the space of all connections. We do not know of a similar geometrical interpretation for any other special value of the torsion.

2 Nearly Kähler coset spaces

2.1 Basic definitions

An $SU(3)$ -structure on a six-manifold is by definition a reduction of the structure group of the tangent bundle to $SU(3)$. Manifolds of dimension six with $SU(3)$ -structure admit a set of canonical objects fixed by $SU(3)$, consisting of an almost complex structure J , a Riemannian metric g , a real two-form ω and a complex three-form Ω . With respect to J , the forms ω and Ω are of type $(1,1)$ and $(3,0)$, respectively, and there is a compatibility condition, $g(J\cdot, \cdot) = \omega(\cdot, \cdot)$. With respect to the volume form V_g of g , ω and Ω are normalised so that

$$\omega \wedge \omega \wedge \omega = 6V_g \quad \text{and} \quad \Omega \wedge \bar{\Omega} = -8iV_g. \quad (2.1)$$

A nearly Kähler six-manifold is an $SU(3)$ -structure manifold such that

$$d\omega = 3\rho \operatorname{Im}\Omega \quad \text{and} \quad d\Omega = 2\rho \omega \wedge \omega \quad (2.2)$$

for some real non-zero constant ρ , proportional to the square of the scalar curvature (if ρ was zero, the manifold would be Calabi-Yau). Nearly Kähler manifolds were first studied by Gray [25], and they solve the Einstein equations with positive cosmological constant. More generally, six-manifolds with $SU(3)$ -structure are classified by their intrinsic torsion, and nearly Kähler manifolds form one particular intrinsic torsion class.

There are only four known examples of compact nearly Kähler six-manifolds, and they are all coset spaces:

$$\begin{aligned} & SU(3)/U(1) \times U(1) , \quad Sp(2)/Sp(1) \times U(1) , \\ & G_2/SU(3) = S^6 , \quad SU(2)^3/SU(2) = S^3 \times S^3 . \end{aligned} \quad (2.3)$$

Here $Sp(1) \times U(1)$ is chosen to be a non-maximal subgroup of $Sp(2)$: if elements of $Sp(2)$ are written as 2×2 quaternionic matrices, then elements of $Sp(1) \times U(1)$ are written $\operatorname{diag}(p, q)$, with $p \in Sp(1)$ and $q \in U(1)$. Also, $SU(2)$ is the diagonal subgroup of $SU(2) \times SU(2) \times SU(2)$. These coset spaces G/H were named 3-symmetric by Wolf and Gray, because the subgroup H is the fixed point set of an automorphism s of G satisfying $s^3 = \operatorname{Id}$ [26, 28].

The 3-symmetry actually plays a fundamental role in defining the canonical structures on the coset spaces. The automorphism s induces an automorphism S of the Lie algebra \mathfrak{g} of G , that is $S : \mathfrak{g} \rightarrow \mathfrak{g}$ is linear and satisfies

$$[SX, SY] = S[X, Y] \quad \forall X, Y \in \mathfrak{g} . \quad (2.4)$$

The cosets under consideration are all reductive, which means that there is a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where \mathfrak{h} is the Lie algebra of H and \mathfrak{m} satisfies $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Actually, on $SU(2)^3/SU(2)$ there is a choice of subspaces \mathfrak{m} ; we choose \mathfrak{m} so that it is orthogonal to \mathfrak{h} with respect to the Cartan-Killing form in this case. The map S acts trivially on \mathfrak{h} and non-trivially on \mathfrak{m} ; one can define a map $J : \mathfrak{m} \rightarrow \mathfrak{m}$ by

$$S|_{\mathfrak{m}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}J = \exp\left(\frac{2\pi}{3}J\right) . \quad (2.5)$$

The map J satisfies $J^2 = -1$ and provides the almost complex structure on G/H .

A natural quadratic form on \mathfrak{m} is given by the Cartan-Killing form of \mathfrak{g} ,

$$\langle X, Y \rangle_{\mathfrak{g}} = -\operatorname{Tr}_{\mathfrak{g}}(\operatorname{ad}(X) \circ \operatorname{ad}(Y)) . \quad (2.6)$$

This extends to a G -invariant metric g on G/H . The $(1,1)$ -form ω is fixed by its compatibility with g and J , and Ω is the unique suitably normalised G -invariant $(3,0)$ -form.

2.2 Lie algebra identities

In calculations, it is useful to choose a basis $\{I_A\}$ for the Lie algebra \mathfrak{g} . We do so in such a way that I_a for $a = 1, \dots, 6$ form a basis for \mathfrak{m} and I_i for $i = 7, \dots, \dim(G)$ yield a basis for \mathfrak{h} . The structure constants f_{AB}^C are defined by

$$[I_A, I_B] = f_{AB}^C I_C \quad \text{with} \quad f_{AC}^D f_{DB}^C = \delta_{AB} , \quad (2.7)$$

where we have chosen the basis so that it is orthonormal with respect to the Cartan-Killing form. Then $f_{ABC} := f_{AB}^D \delta_{DC}$ is totally antisymmetric.

The reductive property of the coset means that the structure constants f_{aij} vanish. The components J_{ab} of the almost complex structure J are defined via $J(I_a) = J_{ab}I_b$. Then the 3-symmetry property (2.4) implies useful identities involving J : notably, the tensor

$$\tilde{f}_{abc} := f_{abd}J_{dc} \quad (2.8)$$

is totally antisymmetric; also

$$J_{cd}f_{adi} = J_{ad}f_{cdi} . \quad (2.9)$$

Another useful identity is

$$J_{ab}f_{abi} = 0 . \quad (2.10)$$

We do not have a general proof of this identity, but we have verified it on each of the four coset spaces. It has the following interpretation: the action of H on \mathfrak{m} defines an embedding of H in $\mathrm{GL}(6, \mathbb{R})$. It is easy to show that H fixes the quadratic form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ and almost complex structure J ; hence H is contained in $\mathrm{U}(3) \subset \mathrm{GL}(6, \mathbb{R})$. The above identity merely asserts that $H \subset \mathrm{SU}(3)$. Geometrically, this means that the natural H -structure on G/H is contained within the $\mathrm{SU}(3)$ -structure.

Apart from $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, there are two other natural quadratic forms on \mathfrak{m} :

$$\langle X, Y \rangle_{\mathfrak{m}} := -\mathrm{Tr}_{\mathfrak{m}}(P_{\mathfrak{m}} \circ \mathrm{ad}(X) \circ P_{\mathfrak{m}} \circ \mathrm{ad}(Y)) , \quad (2.11)$$

$$\langle X, Y \rangle_{\mathfrak{h}} := -\mathrm{Tr}_{\mathfrak{h}}(P_{\mathfrak{h}} \circ \mathrm{ad}(X) \circ P_{\mathfrak{m}} \circ \mathrm{ad}(Y)) , \quad (2.12)$$

where $P_{\mathfrak{m}}$ and $P_{\mathfrak{h}}$ denote the projections onto \mathfrak{m} and \mathfrak{h} , respectively. It is easy to show that

$$\langle \cdot, \cdot \rangle_{\mathfrak{g}} = \langle \cdot, \cdot \rangle_{\mathfrak{m}} + 2 \langle \cdot, \cdot \rangle_{\mathfrak{h}} . \quad (2.13)$$

Furthermore, on the coset spaces in question, one also has

$$\langle \cdot, \cdot \rangle_{\mathfrak{m}} = \frac{1}{3} \langle \cdot, \cdot \rangle_{\mathfrak{g}} . \quad (2.14)$$

Hence, in terms of the structure constants,

$$f_{aci}f_{bci} = f_{acd}f_{bcd} = \frac{1}{3}\delta_{ab} . \quad (2.15)$$

The proof of this identity will be deferred until the end of this section. Note that for three of the four coset spaces this identity has been verified directly in [30].

2.3 Orthonormal frame for the coset

The metric and almost complex structure on \mathfrak{m} lift to a G -invariant metric and almost complex structure on G/H . Local expressions for these can be obtained by introducing an orthonormal frame as follows. The basis elements I_A of the Lie algebra \mathfrak{g} can be represented by left-invariant vector fields \hat{E}_A on the Lie group G , and the dual basis \hat{e}^A is a set of left-invariant one-forms. The space G/H consists of left cosets gH and the natural projection $g \mapsto gH$ is denoted $\pi : G \rightarrow G/H$. Over a contractible open subset U of G/H , one can choose a map $L : U \rightarrow G$ such that $\pi \circ L$ is the

identity (in other words, L is a local section of the principal bundle $G \rightarrow G/H$). The pull-backs of \hat{e}^A by L are denoted e^A . In particular, e^a form an orthonormal frame for $T^*(G/H)$ over U (where again $a = 1, \dots, 6$), and we can write $e^i = e_a^i e^a$ with real functions e_a^i . The dual frame for $T(G/H)$ will be denoted E_a . The forms e^A obey the Maurer-Cartan equations,

$$\begin{aligned} de^a &= -f_{ib}^a e^i \wedge e^b - \frac{1}{2} f_{bc}^a e^b \wedge e^c, \\ de^i &= -\frac{1}{2} f_{bc}^i e^b \wedge e^c - \frac{1}{2} f_{jk}^i e^j \wedge e^k. \end{aligned} \quad (2.16)$$

Since all the connections we consider will be invariant under some action of G , it will suffice to do calculations just over the subset U .

Local expressions for the G -invariant metric, almost complex structure, and nearly Kähler form on G/H are then

$$g = \delta_{ab} e^a e^b, \quad J = J_{ab} e^a E_b, \quad \text{and} \quad \omega = \frac{1}{2} J_{ab} e^a \wedge e^b. \quad (2.17)$$

One can also obtain a local expression for (3,0)-form Ω . From (2.16) one can compute $d\omega$ and hence $*d\omega$:

$$d\omega = -\frac{1}{2} \tilde{f}_{abc} e^a \wedge e^b \wedge e^c \quad \text{and} \quad *d\omega = \frac{1}{2} f_{abc} e^a \wedge e^b \wedge e^c. \quad (2.18)$$

We have that $d\omega = 3\rho \text{Im}\Omega$, and Ω should be normalised so that $\|\text{Im}\Omega\|^2 = 4$. On the other hand, from (2.15) we compute that $\|d\omega\|^2 = 3$. So it must be that $\rho = 1/2\sqrt{3}$ and

$$\text{Im}\Omega = -\frac{1}{\sqrt{3}} \tilde{f}_{abc} e^a \wedge e^b \wedge e^c, \quad \text{Re}\Omega = -\frac{1}{\sqrt{3}} f_{abc} e^a \wedge e^b \wedge e^c. \quad (2.19)$$

Given a pair of differential forms u, v such that the degree of u is less than or equal to the degree of v , their contraction is defined to be

$$u \lrcorner v := *(u \wedge *v). \quad (2.20)$$

If u and v have the same degree, $u \lrcorner v$ coincides with the usual inner product of forms induced by the metric. We are now in a position to prove (2.15): from (2.19), it is equivalent to

$$g(u \lrcorner \text{Re}\Omega, v \lrcorner \text{Re}\Omega) = 2g(u, v) \quad \forall u, v \in \Lambda^1. \quad (2.21)$$

This identity holds on any 6-manifold with $\text{SU}(3)$ -structure, as can be verified by direct calculation in an orthonormal basis.

3 Instantons in six dimensions

3.1 ω -anti-self-duality

Let Ψ be a $(d-4)$ -form on a d -dimensional Riemannian manifold. A natural generalisation of the $d=4$ anti-self-duality equations is the so-called Ψ -anti-self-duality equation,

$$\Psi \wedge \mathcal{F} = -*\mathcal{F}. \quad (3.1)$$

If Ψ is closed, this equation implies the Yang-Mills equation, $D*\mathcal{F} = 0$. Equations of this sort were first written down in [13], using the language of tensors rather than differential forms. They

often have an interpretation as BPS equations, in particular all of the Ψ -anti-self-duality equations considered in this paper are BPS equations.

On a nearly Kähler six-manifold, a natural choice for Ψ is the (1,1)-form ω , giving

$$\omega \wedge \mathcal{F} = - * \mathcal{F} \quad \Leftrightarrow \quad *(\omega \wedge \mathcal{F}) = -\mathcal{F} . \quad (3.2)$$

Of course, ω is not closed, so (3.2) does not imply the Yang-Mills equation, but rather the Yang-Mills equation with torsion,

$$D * \mathcal{F} + d\omega \wedge \mathcal{F} = 0 . \quad (3.3)$$

The ω -anti-self-duality equation (3.2) means that we are looking for eigen-two-forms \mathcal{F} of the operator $*(\omega \wedge \cdot)$, with eigenvalue $\lambda = -1$. The space Λ^2 of two-forms decomposes into three eigenspaces Λ_λ^2 , with the following properties:

λ	2	1	-1
$\dim \Lambda_\lambda^2$	1	6	8
\mathcal{F} -type	$\sim \omega$	$(2,0), (0,2)$	$(1,1) \perp \omega$

(3.4)

Hence, (3.2) is equivalent to the so-called Donaldson-Uhlenbeck-Yau, or Hermitian-Yang-Mills, equations [15]:

$$\mathcal{F}^{0,2} = \mathcal{F}^{2,0} = 0 \quad \text{and} \quad \omega \lrcorner \mathcal{F} = 0 . \quad (3.5)$$

It is interesting to note that when \mathcal{F} solves the ω -anti-self-duality equation (3.2), the torsional term in the Yang-Mills equation (3.3) vanishes, as was pointed out by Xu [31]. This is because \mathcal{F} is a (1,1)-form and $d\omega$ is a sum of (3,0)- and (0,3)-forms: their wedge product then has to vanish.

3.2 Gauge group H

First we consider H -instantons on G/H . The natural projection $G \rightarrow G/H$ defines a principal bundle with structure group H , on which G acts from the left. There is a unique G -invariant connection on this bundle, the so-called canonical connection [32, 33]. On $S^2 = \text{SU}(2)/\text{U}(1)$ the canonical connection is the Dirac monopole and on $S^4 = \text{Sp}(2)/\text{Sp}(1) \times \text{Sp}(1)$ it is the sum of an instanton and an anti-instanton, so it seems a good candidate solution to (3.2) on a nearly Kähler coset space.

In local coordinates, the canonical connection is written

$$\mathcal{A} = e^i I_i = e^a e_a^i I_i . \quad (3.6)$$

Its curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is given in [32], chapter II, theorem 11.1, and is also easily computed using (2.16):

$$\mathcal{F} = -\frac{1}{2} f_{ab}^i e^a \wedge e^b I_i . \quad (3.7)$$

The identity (2.9) implies that this \mathcal{F} is a (1,1)-form, since (1,1)-forms θ are defined by the property $\theta(J\cdot, J\cdot) = \theta(\cdot, \cdot)$. The identity (2.10) tells us that $\omega \lrcorner \mathcal{F} = 0$, where $\omega = \frac{1}{2} J_{ab} e^a \wedge e^b$. So on each of the four nearly Kähler coset spaces, the canonical connection satisfies the Donaldson-Uhlenbeck-Yau equation (3.5), or equivalently the ω -anti-self-duality equation (3.2). The case $G/H = G_2/\text{SU}(3)$ was considered by Xu [31], who also showed that the canonical connection admits no continuous deformation preserving (3.2). In other words, this connected component of the moduli space of solutions consists of just a point.

3.3 Gauge group G

Next, we consider G -instantons on G/H . According to [32] and [34], G -invariant connections with gauge group G are determined by linear maps $\Lambda : \mathfrak{m} \rightarrow \mathfrak{g}$ which commute with the adjoint action of H : $\Lambda(\text{Ad}(h)X) = \text{Ad}(h)\Lambda(X)$, $\forall h \in H$, $X \in \mathfrak{m}$. Such a linear map is represented by a matrix (Φ_{aB}) , such that $\Lambda(I_a) = \Phi_{aB}I_B$, and in local coordinates the connection is written

$$\mathcal{A} = e^i I_i + e^a \Phi_{aB} I_B . \quad (3.8)$$

We make the simple choice

$$\Phi_{ab} = \phi_1 \delta_{ab} + \phi_2 J_{ab} \quad \text{and} \quad \Phi_{ai} = 0 , \quad (3.9)$$

for real numbers ϕ_1 and ϕ_2 , which is more general than the choice considered in [21]. On the space $G_2/\text{SU}(3)$, (3.8) with (3.9) is the most general G_2 -invariant connection, but on the other coset spaces it is not – we will briefly discuss more general choices in the next section. The curvature $\mathcal{F} = \frac{1}{2}\mathcal{F}_{ab}e^a \wedge e^b$ is given in [32], chapter II, theorem 11.7, and can also be computed using (2.16):

$$\mathcal{F}_{ab} = f_{ac}^i (\Phi^\top \Phi - \text{Id})_{cb} I_i + f_{abc} (-\Phi + (\Phi^\top)^2)_{cd} I_d , \quad (3.10)$$

where $(\text{Id}, J)^\top = (\text{Id}, -J)$. Note that \mathcal{F} remembers the 3-symmetry S :

$$\Phi \mapsto \exp(\frac{2}{3}\pi J)\Phi \quad \Rightarrow \quad (-\Phi + (\Phi^\top)^2) \mapsto \exp(\frac{2}{3}\pi J)(-\Phi + (\Phi^\top)^2) . \quad (3.11)$$

The two-forms

$$e^c \lrcorner * d\omega = \frac{3}{2}f_{abc}e^a \wedge e^b \quad \text{and} \quad e^c \lrcorner d\omega = -\frac{3}{2}\tilde{f}_{abc}e^a \wedge e^b \quad (3.12)$$

are clearly of type $(2,0)+(0,2)$, since $d\omega$ is of type $(3,0)+(0,3)$. So this connection solves the ω -anti-self-duality equation (3.2) if and only if

$$-\Phi + (\Phi^\top)^2 = 0 . \quad (3.13)$$

Apart from the canonical connection $\Phi = 0$, the other solutions to this equation are

$$\Phi = \text{Id} , \quad \exp(\frac{2}{3}\pi J) , \quad \exp(\frac{4}{3}\pi J) . \quad (3.14)$$

Note that these connections in fact all have zero curvature.

4 Yang-Mills equations in seven dimensions

4.1 From Yang-Mills theory to a Φ^4 model

On a d -dimensional Riemannian manifold, the Yang-Mills equation with torsion is

$$D * \mathcal{F} + * \mathcal{H} \wedge \mathcal{F} = 0 , \quad (4.1)$$

with \mathcal{H} a three-form. Equation (3.3) is a special case in $d = 6$. We will study solutions of this equation on the seven-dimensional manifolds $\mathbb{R} \times G/H$, with G/H a nearly Kähler coset space. We choose the metric and volume form,

$$g_7 = (e^0)^2 + g_6 \quad \text{and} \quad V_7 = e^0 \wedge V_6 , \quad (4.2)$$

where $e^0 = d\tau$ and τ is a coordinate on \mathbb{R} , while g_6 and V_6 are the metric and volume form on G/H . For \mathcal{H} we make the choice

$$*\mathcal{H} = -\frac{1}{3}\kappa_1 d\tau \wedge d\omega + \frac{1}{3}\kappa_2 *\omega . \quad (4.3)$$

This choice for \mathcal{H} is clearly invariant under the action of G , and under translations in and reversals of τ – in fact, it is the most general possible choice satisfying these conditions. If one does not require invariance under τ -reversals, then a term proportional $d\tau \wedge \omega$ could be added to \mathcal{H} and possibly others, depending on the choice of coset space.

For the connection one-form $\mathcal{A} = \mathcal{A}_0 e^0 + \mathcal{A}_a e^a$, we copy from the previous section the G -invariant ansatz (3.8) and (3.9),

$$\mathcal{A}(\tau) = e^i I_i + e^a \Phi_{ab}(\tau) I_b \quad \text{with} \quad \Phi = \phi_1 \text{Id} + \phi_2 J , \quad (4.4)$$

where ϕ_1 and ϕ_2 are now functions of τ . This ansatz has $\mathcal{A}_0 = 0$, but no generality is lost here since such a gauge can always be chosen. The curvature $\mathcal{F} = \mathcal{F}_{0a} e^0 \wedge e^a + \frac{1}{2} \mathcal{F}_{ab} e^a \wedge e^b$ of this connection has the components (see (3.10))

$$\mathcal{F}_{ab} = f_{ac}^i (\Phi^\top \Phi - \text{Id})_{cb} I_i + f_{abc} (-\Phi + (\Phi^\top)^2)_{cd} I_d \quad \text{and} \quad \mathcal{F}_{0a} = \dot{\Phi}_{ab} I_b , \quad (4.5)$$

where a dot denotes a derivative with respect to τ .

In order to write the Yang-Mills equation in components, it is necessary to introduce the torsionful spin connection on G/H [34]. Recall that a linear connection is a matrix of one-forms $\omega_b^a = e^c \omega_{cb}^a$. The connection is metric compatible if $\omega_a^c g_{cb}$ is anti-symmetric, and its torsion is a vector of two-forms $T^a = \frac{1}{2} T_{bc}^a e^b \wedge e^c$ determined by the structure equation

$$de^a + \omega_b^a \wedge e^b = T^a . \quad (4.6)$$

Our choice is

$$T^a = -e^a \lrcorner \mathcal{H} \quad \Leftrightarrow \quad T_{bc}^a = \kappa_{ad} f_{dbc} \quad \text{with} \quad \kappa := \kappa_1 \text{Id} - \kappa_2 J . \quad (4.7)$$

We take ω_b^a to be the unique metric-compatible linear connection with this torsion. Explicitly,

$$\omega_{cb}^a = e_c^i f_{ib}^a + \frac{1}{2} (\kappa + \text{Id})_{ad} f_{dcb} . \quad (4.8)$$

The torsionful spin connection on $\mathbb{R} \times G/H$ is given by a similar formula, with additional components vanishing:

$$\omega_{0b}^0 = \omega_{0b}^a = \omega_{cb}^0 = 0 . \quad (4.9)$$

Using (4.6), one can show that the Yang-Mills equation with torsion (4.1) is equivalent to

$$E_a \mathcal{F}^{a0} + \omega_{ab}^a \mathcal{F}^{b0} + [\mathcal{A}_a, \mathcal{F}^{a0}] = 0 , \quad (4.10)$$

$$E_0 \mathcal{F}^{0b} + E_a \mathcal{F}^{ab} + \omega_{da}^d \mathcal{F}^{ab} + \omega_{cd}^b \mathcal{F}^{cd} + [\mathcal{A}_a, \mathcal{F}^{ab}] = 0 . \quad (4.11)$$

It is now a matter of computation to substitute the ansatz (4.4) into (4.10) and (4.11), making use of structure constant identities introduced above. One finds that (4.10) is identically satisfied, while (4.11) is equivalent to

$$6\ddot{\Phi} = (\kappa-1)\Phi - (\kappa+3)(\Phi^\top)^2 + 4\Phi^\top\Phi^2. \quad (4.12)$$

For more general choices of connection, equation (4.10) is not automatically solved. For example, in the cases $G/H = \text{SU}(3)/\text{U}(1) \times \text{U}(1)$ and $\text{SU}(2)^3/\text{SU}(2)$, the most general G -invariant connection is parametrised by three complex scalars ψ_1, ψ_2, ψ_3 .³ Equation (4.10) then reads

$$\dot{\psi}_1\bar{\psi}_1 - \dot{\bar{\psi}}_1\psi_1 = \dot{\psi}_2\bar{\psi}_2 - \dot{\bar{\psi}}_2\psi_2 = \dot{\psi}_3\bar{\psi}_3 - \dot{\bar{\psi}}_3\psi_3. \quad (4.13)$$

The general solution to these equations is difficult to find, but one obvious solution is $\psi_1 = \psi_2 = \psi_3$. This returns us to our original ansatz (4.4). More general ansätze will be discussed in a future publication.

4.2 Action

There is an alternative method of deriving (4.12), which involves working with actions rather than equations of motion. Notice that, with $\kappa_2 = 0$, the Yang-Mills equation with torsion (4.1) is the equation of motion for the action

$$S = \int_{\mathbb{R} \times G/H} \text{Tr} \left[\mathcal{F} \wedge * \mathcal{F} + \frac{1}{3} \kappa_1 d\tau \wedge \omega \wedge \mathcal{F} \wedge \mathcal{F} \right]. \quad (4.14)$$

In the Calabi-Yau ($\rho = 0$) limit this action agrees with the standard Yang-Mills action up to a boundary term. Substituting the ansatz (4.4) into this action gives

$$S = \text{Vol}(G/H) \int_{\mathbb{R}} d\tau \text{Tr} \left[2\dot{\Phi}^\top \dot{\Phi} + \widehat{V}(\Phi) \right] \quad \text{with} \quad (4.15)$$

$$3\widehat{V}(\Phi) = (1 - \frac{\kappa_1}{3}) \text{Id} + (\kappa_1 - 1) \Phi^\top \Phi - (1 + \frac{\kappa_1}{3}) (\Phi^3 + (\Phi^\top)^3) + 2(\Phi^\top \Phi)^2. \quad (4.16)$$

The Euler-Lagrange equation for this integral is again (4.12). The matrix-valued potential \widehat{V} is invariant under the action of the S_3 permutation group generated by the 3-symmetry and the conjugation \top .

Let us review what we have done. Starting from the action (4.14), we first substituted an ansatz and then derived an equation of motion. Previously, we derived the equation of motion (4.1) for the action and then substituted the ansatz. There is no reason to expect these two procedures to lead to the same differential equation, unless the ansatz chosen is the most general ansatz invariant under a given symmetry (this is called the principle of symmetry criticality [35]). In our case, the two procedures *did* lead to the same differential equation. However, our ansatz, while G -invariant, is certainly not the most general G -invariant ansatz, except in the case of S^6 . That the two procedures lead to the same differential equation on the other coset spaces could perhaps be attributed to the algebraic similarities between all four coset spaces.

³For $G/H = \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1)$ one has $\psi_1 = \psi_2$; for S^6 and one has $\psi_1 = \psi_2 = \psi_3 = \phi$.

5 Solutions of the Yang-Mills equation

5.1 Critical points of the potential

Throughout this section, we identify the matrix-valued function $\Phi = \phi_1 \text{Id} + \phi_2 J$ with the complex-valued function $\phi = \phi_1 + i\phi_2$ and, likewise, interpret κ as a complex number. Since $\widehat{V}(\Phi)^\top = \widehat{V}(\Phi^\top) = \widehat{V}(\Phi)$ in (4.16), we also define

$$\widehat{V}(\Phi) =: V(\phi) \text{Id} \quad \Rightarrow \quad 3V(\phi) = (1 - \frac{\kappa_1}{3}) + (\kappa_1 - 1)|\phi|^2 - (1 + \frac{\kappa_1}{3})2\text{Re}\phi^3 + 2|\phi|^4. \quad (5.1)$$

If $\kappa_2 = 0$, the equation of motion (4.12) can be written in terms of the real function V ,

$$6\ddot{\phi} = (\kappa - 1)\phi - (\kappa + 3)\bar{\phi}^2 + 4\bar{\phi}\phi^2 = 3\frac{\partial V}{\partial \bar{\phi}}. \quad (5.2)$$

This is the equation of motion of a particle moving in the complex plane under the influence of a potential $-V$. Equation (5.2) admits this mechanical interpretation only when $\kappa_2 = 0$ since, if κ was complex, the potential function V could not be chosen real. In this section, we study solutions of (5.2) with $\kappa_2 = 0$, using this mechanical analogy. We briefly discuss solutions of (4.12) with $\kappa_2 \neq 0$ at the end of the section. Figure 1 displays the equipotential lines of $V(\phi)$ for two special cases.

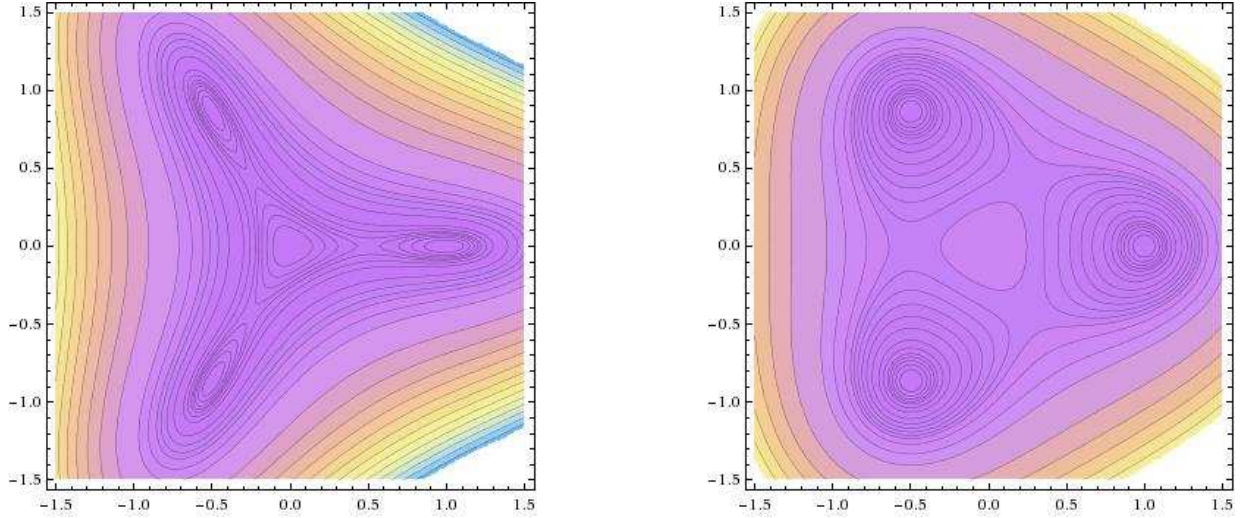


Figure 1: contour plots of the potential $V(\phi)$ for $\kappa = +3$ (left) and for $\kappa = -1$ (right)

We are particularly interested in instantons, which correspond to particle trajectories interpolating between critical points of V , attained at $\tau = \pm\infty$. By conservation of energy, such a trajectory can exist only if $V|_{+\infty} = V|_{-\infty}$. The critical points ϕ^0 of V along the real axis are

ϕ^0	0	1	ν
$V(\phi^0)$	$\frac{2}{9}(1-2\nu)$	0	$\frac{2}{9}(1+\nu)(1-\nu)^3$

with $\nu := \frac{1}{4}(\kappa_1 - 1)$. (5.3)

Since V is invariant under $\phi \mapsto \exp(\frac{2}{3}\pi i)\phi$, for nonzero ϕ^0 there are further critical points $\exp(\frac{2}{3}\pi i)\phi^0$ and $\exp(\frac{4}{3}\pi i)\phi^0$, degenerate in energy with ϕ^0 . At any value of κ_1 , one may therefore search for

trajectories connecting two critical points related by 3-symmetry, which we shall call “transverse”. We will show below that transverse trajectories exist for $\kappa_1 = -7, -1$ (i.e. $\nu = -2, -\frac{1}{2}$). If $\kappa_1 = -3, 3, 9$ (i.e. $\nu = -1, \frac{1}{2}, 2$), two of the critical points on the real axis are degenerate in energy, and one may in addition look for “radial” trajectories connecting them.

The sought-for instanton configurations have finite action only when $V|_{\pm\infty} = 0$. Among the five special cases just mentioned, this occurs for $\kappa_1 = -3, -1, 3$ (i.e. $\nu = -1, -\frac{1}{2}, \frac{1}{2}$). Finite-energy bounce solutions, connecting $\phi^0 = 0$ to itself, may exist for $\kappa_1 < -3$ and for $3 < \kappa_1 < 5$.

κ_1	-7	-3	-1	3	9
ν	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	2
degeneration	none	$V(\exp(i\alpha))$	none	$V(0) = V(1)$	$V(0) = V(2)$
instanton	transverse	radial	transverse	radial	radial
$\phi^0(\pm\infty)$	$\exp(\pm\frac{2}{3}\pi i)(-2)$	$\exp(i\alpha)(\pm 1)$	$\exp(\pm\frac{2}{3}\pi i)(+1)$	$\frac{1}{2} \pm \frac{1}{2}$	1 ± 1
action	infinite	finite	finite	finite	infinite

(5.4)

5.2 Duality

When $\kappa_2 = 0$, there is a surprising duality that relates pairs of values of κ_1 . This is best seen when the equation of motion (5.2) and the potential (5.1) are rewritten in terms of ν :

$$6\ddot{\phi} = 4\nu\phi - 4(\nu+1)\bar{\phi}^2 + 4\bar{\phi}\phi^2 \quad \text{and} \quad 3V(\phi) = \frac{2}{3}(1-2\nu) + 4\nu|\phi|^2 - \frac{8}{3}(1+\nu)\text{Re}\phi^3 + 2|\phi|^4. \quad (5.5)$$

It is straightforward to check that

$$(\nu, \phi(\tau)) \mapsto \left(\frac{1}{\nu}, \frac{1}{\nu}\phi\left(\frac{\tau}{\nu}\right)\right) \quad (5.6)$$

maps solutions of (5.2) to other solutions. We do not know the origin of this duality.

5.3 Gradient flow

We discuss here the case $(\kappa_1, \kappa_2) = (3, 0)$. Our discussion applies also to $(\kappa_1, \kappa_2) = (9, 0)$, via the duality transformation. The potential V can be written in terms of a real “superpotential” W :

$$3V = 2\left|\frac{\partial W}{\partial \phi}\right|^2 \quad \text{with} \quad W = \frac{1}{3}(\phi^3 + \bar{\phi}^3) - |\phi|^2. \quad (5.7)$$

So (5.2) is implied by the gradient flow equation

$$\pm\sqrt{3}\dot{\phi} = \bar{\phi}^2 - \phi = \frac{\partial W}{\partial \bar{\phi}}. \quad (5.8)$$

Finite-action kink solutions are

$$\phi(\tau) = \frac{1}{2}(1 \pm \tanh(\frac{\tau-\tau_0}{2\sqrt{3}})) , \quad (5.9)$$

with τ_0 being the collective coordinate. Further solutions are obtained by applying the 3-symmetry. Since W is 3-symmetric, it is clear that the gradient flow (which reduces the value of W along a path) does not have any transverse solutions.

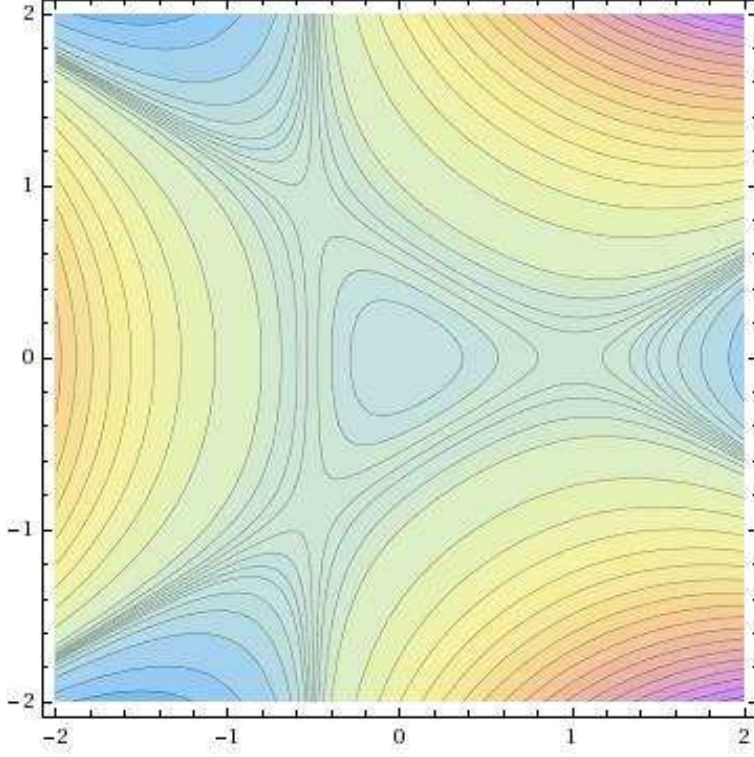


Figure 2: contour plot of the superpotential $W(\phi)$ for $\kappa = +3$ (and for $\kappa = -1$)

We have also found explicit infinite-action solutions of the gradient flow equations: if we define two real functions $r(\tau)$ and $\varphi(\tau)$ by the polar decomposition $\phi = r \exp(i\varphi)$, then (5.8) is equivalent to

$$\sqrt{3}\dot{r} = r - r^2 \cos 3\varphi \quad \text{and} \quad \sqrt{3}\dot{\varphi} = r \sin 3\varphi . \quad (5.10)$$

It follows that

$$\frac{dr}{d\varphi} = \operatorname{cosec} 3\varphi - r \cot 3\varphi , \quad (5.11)$$

assuming $\dot{\varphi} \neq 0$. This can be integrated using a standard formula to give

$$r(\varphi) = -\frac{1}{3} (\sin 3\varphi)^{-1/3} \left[\cos 3\varphi {}_2F_1\left(\frac{1}{2}, \frac{5}{6}, \frac{3}{2}, \cos^2 3\varphi\right) + C \right] \quad (5.12)$$

for some real integration constant C . The hypergeometric function ${}_2F_1$ arises from the antiderivative of $(\sin 3\varphi)^{-2/3}$. Since $r(\varphi)$ diverges for $3\varphi = n\pi$, the trajectories are unbounded. However, this solution does not capture the special case of radial motion:

$$\dot{\varphi} = 0 \quad \Leftrightarrow \quad 3\varphi = n\pi \quad \text{and} \quad \sqrt{3}\dot{r} = r(1-r) , \quad (5.13)$$

which yields our previous kinks, moving radially in the special directions.

5.4 Continuous symmetry

The case $(\kappa_1, \kappa_2) = (-3, 0)$ is special: firstly, because it is fixed by the duality transformation; and secondly, because the potential function V is invariant under not only the 3-symmetry, but also

under U(1) rotations of ϕ . Again, we can find a superpotential W ,

$$3V = 2 \left| \frac{\partial W}{\partial \bar{\phi}} \right|^2 \quad \text{with} \quad W = \frac{2}{3} |\phi|^3 - 2|\phi|. \quad (5.14)$$

Like before, solutions of the gradient flow equation

$$\pm \sqrt{3} \dot{\phi} = \frac{\phi}{|\phi|} (1 - |\phi|^2) = \frac{\partial W}{\partial \bar{\phi}} \quad (5.15)$$

solve (5.2). However, care should be taken near the origin, where W is not differentiable. The finite-action solutions are

$$\phi(\tau) = \pm \tanh\left(\frac{\tau - \tau_0}{\sqrt{3}}\right) \quad (5.16)$$

and U(1) rotations of these. The only other solutions are infinite-action trajectories connecting the critical circle $|\phi| = 1$ with $\phi = \infty$:

$$\phi(\tau) = \mp \coth\left(\frac{\tau - \tau_0}{\sqrt{3}}\right), \quad (5.17)$$

modulo U(1) rotations. In particular, there are no transverse solutions of the gradient flow equation.

5.5 Hamiltonian flow

Now we consider transverse trajectories. Without loss of generality, we look for transverse trajectories connecting $\exp(-\frac{2}{3}\pi i)\phi^0$ and $\exp(\frac{2}{3}\pi i)\phi^0$ with ϕ^0 a non-zero real critical point, i.e. $\phi^0 = 1$ or $\phi^0 = \nu$. For simplicity, we assume that ϕ_1 is constant along these trajectories. Then ϕ_1 and κ should be chosen so that $\ddot{\phi}_1 = 0$ for all ϕ_2 . Since

$$6\ddot{\phi}_1 = [(\kappa_1 - 1)\phi_1 - (\kappa_1 + 3)\phi_1^2 + 4\phi_1^3] + \kappa_2(2\phi_1 + 1)\phi_2 + [(\kappa_1 + 3) + 4\phi_1]\phi_2^2, \quad (5.18)$$

we get exactly three solutions for $\kappa_2 = 0$:

$$(\phi_1, \kappa_1) = (0, -3), \left(-\frac{1}{2}, -1\right), (1, -7). \quad (5.19)$$

We will return to $\kappa_2 \neq 0$ solutions momentarily.

The case $\kappa_1 = -3$ was treated above and yields radial trajectories. The cases $\kappa_1 = -1$ and $\kappa_1 = -7$ are related by the duality transformation, so we consider here just $(\kappa_1, \kappa_2) = (-1, 0)$. Then (5.2) is implied by a first-order hamiltonian flow,

$$\pm \sqrt{3} \dot{\phi} = i(\bar{\phi}^2 - \phi) = i \frac{\partial W}{\partial \bar{\phi}} \quad \text{with} \quad W = \frac{1}{3}(\phi^3 + \bar{\phi}^3) - |\phi|^2, \quad (5.20)$$

i.e. the hamiltonian is exactly the superpotential of (5.7). There are finite-action solutions

$$\phi(\tau) = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2} \tanh\left(\frac{\tau - \tau_0}{2}\right). \quad (5.21)$$

Two further solutions are obtained on application of the 3-symmetry.

It is straightforward to write down infinite-action solutions of (5.20), at least in implicit form. The value of W is conserved by the flow, so solutions $\phi(\tau)$ of the flow obey $W(\phi(\tau)) = C$ for some constant C . In polar coordinates $\phi = r \exp(i\varphi)$, this reads

$$\cos 3\varphi = \frac{3}{2} \frac{r^2 + C}{r^3}, \quad (5.22)$$

which yields $r(\varphi)$ as a solution of a cubic equation. In particular, for $-1/3 < C < 0$ there are periodic trajectories. It is amusing to note that the gradient-flow and hamiltonian-flow cases are related by flipping the sign of ν .

5.6 Solutions without action

Referring again to equation (5.18) we see that, if $\kappa_2 \neq 0$, then $\ddot{\phi}_1 = 0$ enforces $\phi_1 = -\frac{1}{2}$, hence the case

$$(\phi_1, \kappa_1) = (-\frac{1}{2}, -1) \quad (5.23)$$

of (5.19) allows us to turn on κ_2 . With these values fixed, (4.12) is equivalent to

$$6\ddot{\phi}_2 = 4(\phi_2 - \frac{\sqrt{3}}{2})(\phi_2 + \frac{\sqrt{3}}{2})(\phi_2 - \frac{\kappa_2}{4}). \quad (5.24)$$

This equation has kink-type solutions whenever the roots of the polynomial on the right hand side are evenly spaced. This occurs not only in the case $\kappa_2 = 0$ discussed above, but also when $\kappa_2 = \pm 6\sqrt{3}$. The corresponding kink solutions are

$$\phi = -\frac{1}{2} + \epsilon i \left(\frac{\sqrt{3}}{2} \pm \sqrt{3} \tanh(\tau - \tau_0) \right) \quad \text{with} \quad \epsilon = \text{sgn}(\kappa_2). \quad (5.25)$$

6 Instanton equations in seven and eight dimensions

6.1 Anti-self-duality in eight dimensions

In the previous section we constructed solutions of the Yang-Mills equation on $G/H \times \mathbb{R}$ for special values of κ . We found that the second-order Yang-Mills equations actually reduced to first-order equations for these special values. In this section we will show that those first-order equations which admit finite-energy instantons have a natural geometrical interpretation: they take the anti-self-duality form (3.1) with a suitably chosen three-form Ψ .

It is most convenient to start in eight dimensions rather than in seven. Let x^7 and x^8 denote coordinates on \mathbb{R}^2 and let $e^7 = dx^7$ and $e^8 = dx^8$; then the forms

$$\tilde{\omega} = \omega + e^7 \wedge e^8 \quad \text{and} \quad \tilde{\Omega} = \Omega \wedge (e^7 + ie^8) \quad (6.1)$$

define an $\text{SU}(4)$ -structure on $G/H \times \mathbb{R}^2$. The associated metric and volume form are

$$g_8 = g_6 + (e^7)^2 + (e^8)^2 \quad \text{and} \quad V_8 = V_6 \wedge e^7 \wedge e^8. \quad (6.2)$$

The four-form

$$\Sigma = \frac{1}{2} \tilde{\omega} \wedge \tilde{\omega} - \text{Re} \tilde{\Omega} \quad (6.3)$$

defines a Spin(7)-structure. The operator $*_8(\Sigma \wedge \cdot)$ on two-forms has eigenvalues -1 and 3, with eigenspaces of dimensions 21 and 7, respectively. So it makes sense to consider the Σ -anti-self-duality equation

$$\Sigma \wedge \mathcal{F} = - *_8 \mathcal{F}. \quad (6.4)$$

This equation has been studied in [13, 4, 3, 10]. With respect to a complex basis

$$\{\Theta^\alpha\}: \quad \Theta^1 = e^1 + ie^2, \quad \Theta^2 = e^3 + ie^4, \quad \Theta^3 = e^5 + ie^6, \quad \Theta^4 = e^7 + ie^8 \quad (6.5)$$

it reads

$$\tilde{\omega} \lrcorner \mathcal{F} = 0 \quad \text{and} \quad \mathcal{F}_{\bar{\alpha}\bar{\beta}} = -\frac{1}{2}\epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}\mathcal{F}^{\bar{\gamma}\bar{\delta}} \quad (6 \text{ real equations}), \quad (6.6)$$

where we have raised indices using the almost Hermitian metric: $\mathcal{F}^{\bar{\alpha}\bar{\beta}} = \mathcal{F}_{\alpha\beta} g^{\alpha\bar{\alpha}} g^{\beta\bar{\beta}}$ with $g^{\alpha\bar{\alpha}} = \delta^{\alpha\bar{\alpha}}$. For $\mathcal{A}_7 = \mathcal{A}_8 = 0$ the equations (6.6) reduce to

$$(\partial_7 + i\partial_8)\mathcal{A}_{\bar{p}} = \epsilon_{\bar{p}\bar{q}\bar{r}}\mathcal{F}^{\bar{q}\bar{r}} \quad \text{for } p, q, r = 1, 2, 3, \quad (6.7)$$

whose stable points satisfy $\mathcal{F}^{0,2} = 0$. In the real basis, (6.6) read

$$\partial_7 \mathcal{A}_a - J_{ab} \partial_8 \mathcal{A}_b = \sqrt{3} f_{abc} \mathcal{F}_{bc}. \quad (6.8)$$

6.2 Gradient flow

Now we step down to seven dimensions. Consider the seven-manifold $G/H \times \mathbb{R}$, with \mathbb{R} parametrised by x^7 , the metric g_7 induced from g_8 and the volume form $V_7 = V_6 \wedge e^7$. Then our four-form Σ descends as follows,

$$\Sigma = \Xi \wedge e^8 + *_7 \Xi, \quad (6.9)$$

where

$$\Xi = \omega \wedge e^7 + \text{Im}\Omega \quad \text{and} \quad *_7 \Xi = \frac{1}{2} \omega \wedge \omega - \text{Re}\Omega \wedge e^7 \quad (6.10)$$

live on $G/H \times \mathbb{R}$. The three-form Ξ defines a G_2 -structure, which is compatible with the metric in the sense that $*_7(i_u \Xi \wedge i_v \Xi \wedge \Xi) = 6 g_7(u, v)$ for all tangent vectors u, v [36].

Associated to Ξ is an anti-self-duality equation (3.1). The eigenvalue problem for the operator $*_7(\Xi \wedge \cdot)$ on two-forms \mathcal{F} is characterised as follows,

λ	2	-1
$\dim \Lambda_\lambda^2$	7	14
\mathcal{F} -type	$i_u \Xi$	$*_7 \Xi \wedge \mathcal{F} = 0$

(6.11)

The space Λ_{-1}^2 maps to the Lie algebra of G_2 under the isomorphism $\Lambda^2 \cong \mathfrak{so}(7)$.

Now suppose that \mathcal{F} is a connection on $G/H \times \mathbb{R}^2$ pulled back from $G/H \times \mathbb{R}$ or, equivalently, that $\mathcal{A}_8 = 0$ and $\mathcal{A}_1, \dots, \mathcal{A}_7$ are independent of x^8 in some gauge. Then it is easy to show that

$$\Sigma \wedge \mathcal{F} = \Xi \wedge \mathcal{F} \wedge e^8 + *_7 \Xi \wedge \mathcal{F} \quad \text{and} \quad *_8 \mathcal{F} = *_7 \mathcal{F} \wedge e^8. \quad (6.12)$$

Hence, the Σ -anti-self-duality (6.4) in eight dimensions descends to

$$\Xi \wedge \mathcal{F} = - *_7 \mathcal{F}. \quad (6.13)$$

This equation was studied in detail in [8]. Differentiating, one sees that this equation implies the Yang-Mills equation (4.1) with torsion given by $(\kappa_1, \kappa_2) = (3, 0)$, on identifying $\tau = x^7$. Further, a solution of the anti-self-duality equation (3.2) on G/H pulls back to a τ -independent solution of (6.13).

Let us rewrite (6.13) in components. This is easily done using the fact that (6.13) is equivalent to $(i_u \Xi) \lrcorner \mathcal{F} = 0$ for tangent vectors $u = E_1, \dots, E_7$. With

$$i_{E_7} \Xi = -\frac{1}{2} J_{ab} e^a \wedge e^b \quad \text{and} \quad i_{E_a} \Xi = -J_{ab} e^b \wedge e^7 + \sqrt{3} \tilde{f}_{abc} e^b \wedge e^c, \quad (6.14)$$

(6.13) is equivalent to

$$J_{ab} \mathcal{F}_{ab} = 0 \quad \text{and} \quad \mathcal{F}_{7a} = \sqrt{3} \tilde{f}_{abc} \mathcal{F}_{bc}. \quad (6.15)$$

The second relation is the flow equation introduced in [21]. We have shown in Section 3 that the first relation is satisfied by our ansatz (4.4); substituting this ansatz into the second relation yields precisely the gradient flow equation (5.8) for $\kappa = 3$.

6.3 Hamiltonian flow

We now repeat the discussion of the previous subsection, but with the roles of x^7 and x^8 reversed. We regard x^8 as a coordinate on the factor \mathbb{R} of $G/H \times \mathbb{R}$, denote by g'_7 the metric induced from g_8 and choose the volume form $V'_7 = V_6 \wedge e^8$. Then

$$\Sigma = -\Xi' \wedge e^7 + *_7' \Xi', \quad (6.16)$$

where

$$\Xi' = \omega \wedge e^8 + \text{Re} \Omega \quad \text{and} \quad *_7' \Xi' = \frac{1}{2} \omega \wedge \omega + \text{Im} \Omega \wedge e^8. \quad (6.17)$$

Again, Ξ' defines a G_2 -structure on $G/H \times \mathbb{R}$, and the action of $*'_7(\Xi' \wedge \cdot)$ exactly mirrors that of $*_7(\Xi \wedge \cdot)$. In particular, if \mathcal{F} is a connection on $G/H \times \mathbb{R}$ pulled back to $G/H \times \mathbb{R}^2$, then (6.4) is equivalent to

$$\Xi' \wedge \mathcal{F} = -*_7' \mathcal{F}. \quad (6.18)$$

Which second-order equation is implied by (6.18)? Differentiating, one obtains

$$D *_7' \mathcal{F} + (3\rho \text{Im} \Omega \wedge e^8 + 2\rho \omega \wedge \omega) \wedge \mathcal{F} = 0. \quad (6.19)$$

Taking into account the equivalence of (6.18) and $*'_7 \Xi' \wedge \mathcal{F} = 0$, one arrives at the Yang-Mills equation (4.1) with torsion for $(\kappa_1, \kappa_2) = (-1, 0)$.

The component form of (6.18) is

$$J_{ab} \mathcal{F}_{ab} = 0 \quad \text{and} \quad \mathcal{F}_{8a} = -\sqrt{3} \tilde{f}_{abc} \mathcal{F}_{bc}. \quad (6.20)$$

With the ansatz (4.4), the first equation is again automatically satisfied, while the second one is exactly equivalent to the hamiltonian flow equation (5.20) with $\kappa = -1$.

6.4 Continuous symmetry

The fixed points of the gradient flow or hamiltonian flow equations are the critical points of the superpotential W . For the special $U(1)$ -symmetric case $(\kappa_1, \kappa_2) = (-3, 0)$, the superpotential (5.14) yields the fixed points $|\phi|^2 = 1$. From the discussion in Section 3 it is clear that these are the solutions of the ω -self-duality equation,

$$*_6 \mathcal{F} = \omega \wedge \mathcal{F} , \quad (6.21)$$

for a G -invariant connection on G/H . Indeed, differentiating this equation gives the Yang-Mills equation (4.1) with torsion via $(\kappa_1, \kappa_2) = (-3, 0)$. This simple geometrical interpretation for the static solutions does not seem to extend to the general solutions of the flow equation (5.8) in this situation. Note that the ω -self-dual equation is not a BPS equation.

6.5 First-order flows

We close with a second interpretation of the G_2 -instanton equations (6.13) and (6.18), which accounts for the appearance of a superpotential with gradient and hamiltonian flows in section 5.

In his thesis [31], Xu noted that the ω -anti-self-duality equation (3.2) is equivalent to

$$d\omega \wedge \mathcal{F} = 0 . \quad (6.22)$$

We have already seen that (3.2) implies (6.22). To show the converse, we first observe that (6.22) implies $\mathcal{F}^{0,2} = \mathcal{F}^{2,0} = 0$. It follows that

$$\text{Re}\Omega \wedge \mathcal{F} = 0 . \quad (6.23)$$

Second, differentiating and applying the Bianchi identity yields

$$\omega \wedge \omega \wedge \mathcal{F} = 0 \quad \Leftrightarrow \quad \omega \lrcorner \mathcal{F} = 0 . \quad (6.24)$$

Thus, \mathcal{F} contains only a (1,1) part orthogonal to ω . Since the Donaldson-Uhlenbeck-Yau equations (3.5) are equivalent to (3.2), this proves the assertion.

Xu then introduced an action

$$\int_{X^6} \text{Tr}(\omega \wedge \mathcal{F} \wedge \mathcal{F}) \quad (6.25)$$

for a connection on a nearly Kähler six-manifold X^6 , whose equation of motion is (6.22) and whose gradient flow equation is

$$\frac{\partial \mathcal{A}}{\partial \tau} = *(\mathcal{F} \wedge d\omega) . \quad (6.26)$$

Given a local orthonormal frame e^a for the cotangent bundle of X^6 , we contract both sides with the e^a . Employing the identity

$$*(\mathcal{F} \wedge d\omega) \lrcorner e^a = (*d\omega) \lrcorner (\mathcal{F} \wedge e^a) , \quad (6.27)$$

one sees that the flow equation is equivalent to

$$\frac{\partial \mathcal{A}_a}{\partial \tau} = \frac{3}{2} f_{abc} \mathcal{F}_{bc} , \quad (6.28)$$

which coincides with the second equation of (6.15) after a rescaling in τ .

Thus, the G_2 -instanton equation (6.13) implies the gradient flow for the action (6.25). This explains why (6.13) reduces to a gradient flow equation for ϕ . Substituting the ansatz (4.4) into (6.25) should give something proportional to the superpotential (5.7), and this is easily verified:

$$\begin{aligned} 3 \int_{G/H} \text{Tr}(\omega \wedge \mathcal{F} \wedge \mathcal{F}) &= -\text{Vol}(G/H) \text{Tr} \left[1 - 3\Phi^\top \Phi + \Phi^3 + (\Phi^\top)^3 \right] \\ &= -\text{Vol}(G/H) (1 + 3W) . \end{aligned} \quad (6.29)$$

Similarly, the hamiltonian flow equation for (6.25) is

$$\frac{\partial \mathcal{A}_a}{\partial \sigma} J_{ab} e^b = *(\mathcal{F} \wedge d\omega) . \quad (6.30)$$

This is equivalent to the second equation of (6.20), and reduces to the hamiltonian flow for W .

Finally, we comment on the relation between the gradient and hamiltonian flow equations and the remaining part of the G_2 -instanton equations, $\omega \lrcorner \mathcal{F} = 0$ (the first equation in (6.15) or (6.20)). By employing the identities $J * \partial_\tau \mathcal{A} = \frac{1}{2} \partial_\tau \mathcal{A} \wedge \omega \wedge \omega$ and $J(\text{Im} \Omega \wedge \mathcal{F}) = \text{Re} \Omega \wedge \mathcal{F}$, the gradient flow (6.26) can be rearranged to read

$$\frac{1}{2} \frac{\partial \mathcal{A}}{\partial \tau} \wedge \omega \wedge \omega = 3\rho \text{Re} \Omega \wedge \mathcal{F} . \quad (6.31)$$

By taking the exterior derivative, and using the fact that $D(\partial_\tau \mathcal{A}) = \partial_\tau \mathcal{F}$, this equation implies

$$\frac{\partial}{\partial \tau} (\omega \lrcorner \mathcal{F}) = 12\rho^2 \omega \lrcorner \mathcal{F} . \quad (6.32)$$

A similar argument shows that, for the hamiltonian flow (6.30), $\partial_\tau (\omega \lrcorner \mathcal{F}) = 0$. So, we should not be surprised that $\omega \lrcorner \mathcal{F} = 0$ holds for our gradient and hamiltonian flows: if one requires this to hold at $\tau = \pm\infty$, it will hold everywhere.

It is a curious fact that a nearly Kähler structure can itself be regarded as a critical point of a hamiltonian flow [37] – perhaps there is some connection between this and the gradient and hamiltonian flows described above.

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